Multi-Resolution Bayesian Compressive Sensing for Sparse Target Inversion

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1 Basic Formulation

Let us consider an inaccessible investigation domain Λ irradiated by a set of incident transverse-magnetic planes $E_{inc}^{v}(\mathbf{r}^{v})$, v = 1, ..., V, impinging from the angular directions $\theta^{v} = \frac{2\pi}{V}(v-1)$, being V the number of views. In this working scenario, the scattered field $E_{scatt}^{v}(\mathbf{r}_{s}^{v})$, s = 1, ..., S, is supposed to be measured through a set of S sensors equally displaced on a circular observation domain Θ , external to the investigation domain $(\Lambda \cap \Theta = 0)$, having radius ρ . The exact location of the sensors are identified by the position vector $\mathbf{r}_{s}^{v} = (\rho \cos \theta_{s}^{v} \sin \theta_{s}^{v})$, being $\theta_{s}^{v} = \theta^{v} + \frac{2\pi}{S}(s-1)$. This scattered field is known to be dependent on the equivalent currents $J_{eq}^{v}(\mathbf{r})$ generated in the support of the unknown scatterers placed into the domain Λ , according to the *data equation*

$$E_{scatt}^{v}\left(\mathbf{r}_{s}^{v}\right) = -k_{0}^{2} \int_{\Lambda} J_{eq}^{v}\left(\mathbf{r}'\right) G\left(\mathbf{r}_{s}^{v}/\mathbf{r}'\right) \tag{1}$$

where $G(\mathbf{r}_{s}^{v}/\mathbf{r}')$ is the Green's function in the fre space and $k_{0} = \omega \sqrt{\varepsilon_{0}\mu_{0}}$. The material properties of the investigation domain Λ in terms of relative dielectric permittivity $\varepsilon_{r}(\mathbf{r})$ and electric conductivity $\sigma(\mathbf{r})$ are described by means of the object function $\tau(\mathbf{r}) = \varepsilon_{r}(\mathbf{r}) - \varepsilon_{0} - \frac{\sigma(\mathbf{r})}{2\pi f \varepsilon_{0}}$, being f the frequency of the TM plane wave. In order to numerically deal with (1), the investigation domain is discretized into N sub-domains (cells) by means of the Richmond's procedure, providing the matrix form of $\mathbf{E}^{v} = \mathbf{GJ}_{eq}^{v} + \mathbf{N}^{v}$, being G the Green's matrix and \mathbf{N}^{v} the zero mean additive gaussian noise vector of variance σ^{2} . The dielectric features of the N sub-domains described through the discretized form of the object function τ , are then retrieved through the following iterative strategy which combines a multi-resolution approach and the BCSmethod, aimed to maximize the a-posteriori probability of the equivalent souces given the scattered field as:

$$\widehat{\mathbf{J}}_{eq}^{v} = \arg\left\{\max\left[\mathcal{P}\left(\left.\mathbf{J}_{eq}^{v}\right| \mathbf{E}_{scatt}^{v}\right)\right]\right\}, \qquad v = 1, ..., V$$
(2)

More in detail, the algorithms works as follows:

- Initialization: Definition of input parameters of the BCS problem, namely the initial estimation of the noise on the scattered data, σ²_{init}, and the convergence parameter, γ, and the parameter related to the stopping criterion of the IMSA, χ. Set the region of interest equal to the whole domain D⁽¹⁾ = Λ;
- 2. BCS inversion via "Constrained-RVM":
 - (a) increase of the iteration index: i = i + 1;

(b) solution of the BCS problem within the Region of Interest (RoI) $\mathcal{D}^{(i-1)}$ defined at the (i-1)-th step, by maximizing the following cost function:

$$\ell\left(\mathbf{a}^{v}\right) = -0.5\left[2S\log\left(2\pi\right) + \log\left(\mathbf{C}\right) + \left(\mathbf{E}_{scatt}^{v}\right)^{T}\mathbf{C}^{-1}\left(\mathbf{E}_{scatt}^{v}\right)\right], \qquad v = 1, ..., V$$
(3)

where $\mathbf{C} = \sigma^2 \mathbf{I} + \mathbf{G} \left[diag \left(\mathbf{a}^v \right) \right]^{-1} \mathbf{G}^T$ and being \mathbf{a}^v the hyperparameter vector whose entries corresponding to the cells out of the *Rol* $\mathcal{D}^{(i-1)}$ are forced to ∞ ;

3. Equivalent Current Retrieval:

Computation of the equivalent currents starting from the hyperparameter vector \mathbf{a}^{v} according to:

$$\mathbf{J}_{eq}^{v} = \frac{1}{\sigma^{2}} \left[\frac{\mathbf{G}^{T} \mathbf{G}}{\sigma^{2}} diag\left(\mathbf{a}^{v}\right) \right]^{-1} \mathbf{G}^{T} \mathbf{E}_{scatt}^{v}, \qquad v = 1, ..., V$$
(4)

4. Features' Retrieval:

Reconstruction of the material properties of the investigation domain taking advantage from the first order Born approximation through

$$\tau\left(\mathbf{r}_{n}^{(i)}\right) = \frac{1}{V} \sum \frac{\mathbf{J}_{eq}^{v}\left(\mathbf{r}_{n}^{(i)}\right)}{\mathbf{E}_{inc}^{v}\left(\mathbf{r}_{n}^{(i)}\right) + \sum_{p=1}^{N} \mathbf{J}_{eq}^{v}\left(\mathbf{r}_{n}^{(i)}\right) \left[\int_{D} \mathbf{Gr}\right]}, \qquad n = 1, \dots, N$$
(5)

being $\mathbf{r}_n^{(i)}$ the barycenter of the n-th cell within the *Rol* $\mathcal{D}^{(i-1)}$;

5. Convergence Check:

Definition of the new *RoI* $\mathcal{D}^{(i)}$ according to the contrast function distribution calculated with (5) and evaluation of the following termination condition:

$$\left(\frac{L^{(i-1)} - L^{(i)}}{L^{(i)}}\right) < \chi \tag{6}$$

being $L^{(i)}$ the side of the *RoI* $\mathcal{D}^{(i)}$. If (6) is met, then stop the iterative process otherwise go to step 2.

2 Analysis with Different Object Shapes - Domain $L = 6.0\lambda$

2.1 E-shaped Object, $\ell = 1.5\lambda$

Test Case Description

Direct solver:

- Side of the investigation domain: $L = 6.0\lambda$
- Cubic domain divided in $\sqrt{D} \times \sqrt{D}$ cells
- Number of cells for the direct solver: D = 1600 (discretization = $\lambda/10$)

Investigation domain:

- Cubic domain divided in $\sqrt{N} \times \sqrt{N}$ cells
- Number of cells for the inversion:
 - First Step IMSA: $N^{(1)} = 100$ (discretization = $\lambda/10$)
 - Following Steps IMSA: $N^{(i)}$ not fixed, defined according to the estimated $Rol \ \mathcal{D}^{(i)}$

Measurement domain:

- Total number of measurements: M = 60
- Measurement points placed on circles of radius $\rho=4.5\lambda$

Sources:

- Plane waves
- Number of views: $V = 60; \theta_{inc}^v = 0 + (v 1) \times (360/V)$
- Amplitude: A = 1.0
- Frequency: $F = 300 \text{ MHz} (\lambda = 1)$

Background:

- $\varepsilon_r = 1.0$
- $\sigma = 0$ [S/m]

Scatterer

- E-shaped object, $\ell = 1.5\lambda$
- $\varepsilon_r \in \{1.05, 1.10, 1.15, 1.20, 1.50, 2.00, 2.50\}$
- $\sigma = 0$ [S/m]





Figure 1: *E-shaped Object*, $\tau = 0.10$ - (*a*) Actual profile and (*b*)-(*o*) IMSA-BCS reconstructed profiles for (*b*)(*e*)(*h*)(*m*) SNR = 20 [dB], (*c*)(*f*)(*i*)(*n*) SNR = 10 [dB] and (*d*)(*g*)(*l*)(*o*) SNR = 5 [dB] at the step (*b*)-(*d*) S = 1, (*e*)-(*g*) S = 2, (*h*)-(*l*) S = 3 and (*m*)-(*o*) S = 4.

		SNR =	= 50 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	7.65×10^{-3}	4.04×10^{-3}	3.14×10^{-3}	2.38×10^{-3}
ξ_{int}	5.25×10^{-2}	4.12×10^{-2}	3.22×10^{-2}	2.57×10^{-2}
ξ_{ext}	5.29×10^{-3}	2.17×10^{-3}	1.69×10^{-3}	1.20×10^{-3}
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	7.73×10^{-3}	4.08×10^{-3}	3.15×10^{-3}	2.50×10^{-3}
ξ_{int}	5.36×10^{-2}	4.17×10^{-2}	3.20×10^{-2}	2.61×10^{-2}
Eert	5.31×10^{-3}	2.21×10^{-3}	1.71×10^{-3}	1.31×10^{-3}
30.40		_		
3020		SNR =	= 10 <i>dB</i>	
30.00	S = 1	SNR = S = 2	= 10 dB $S = 3$	S = 4
ξtot	S = 1 7.85 × 10 ⁻³	$SNR = $ $S = 2$ 4.16×10^{-3}	= $10dB$ S = 3 3.15×10^{-3}	S = 4 2.27 × 10 ⁻³
ξtot ξint	$S = 1 7.85 \times 10^{-3} 5.37 \times 10^{-2}$	$SNR = \frac{SNR}{S} = 2$ $\frac{4.16 \times 10^{-3}}{4.14 \times 10^{-2}}$	= $10dB$ S = 3 3.15×10^{-3} 3.12×10^{-2}	$S = 4 2.27 \times 10^{-3} 2.30 \times 10^{-2}$
$\frac{\xi_{tot}}{\xi_{int}}$	S = 1 7.85 × 10 ⁻³ 5.37 × 10 ⁻² 5.39 × 10 ⁻³	$SNR = \frac{S = 2}{4.16 \times 10^{-3}}$ $\frac{4.14 \times 10^{-2}}{2.24 \times 10^{-3}}$	= 10dB S = 3 3.15×10^{-3} 3.12×10^{-2} 1.73×10^{-3}	S = 4 2.27 × 10 ⁻³ 2.30 × 10 ⁻² 1.20 × 10 ⁻³
$\frac{\xi_{tot}}{\xi_{int}}$ ξ_{ext}	S = 1 7.85 × 10 ⁻³ 5.37 × 10 ⁻² 5.39 × 10 ⁻³	$SNR = SNR = S = 2$ 4.16×10^{-3} 4.14×10^{-2} 2.24×10^{-3} $SNR = SNR = SNR$	= 10dB S = 3 3.15×10^{-3} 3.12×10^{-2} 1.73×10^{-3} = 5dB	S = 4 2.27 × 10 ⁻³ 2.30 × 10 ⁻² 1.20 × 10 ⁻³
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{ext}}$	S = 1 7.85 × 10 ⁻³ 5.37 × 10 ⁻² 5.39 × 10 ⁻³ $S = 1$	SNR = SNR	$= 10dB$ $S = 3$ 3.15×10^{-3} 3.12×10^{-2} 1.73×10^{-3} $= 5dB$ $S = 3$	S = 4 2.27 × 10 ⁻³ 2.30 × 10 ⁻² 1.20 × 10 ⁻³ $S = 4$
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{tot}}$	S = 1 7.85 × 10 ⁻³ 5.37 × 10 ⁻² 5.39 × 10 ⁻³ $S = 1$ 8.38 × 10 ⁻³	$SNR = \frac{S = 2}{4.16 \times 10^{-3}}$ $\frac{4.14 \times 10^{-2}}{2.24 \times 10^{-3}}$ $SNR = \frac{SNR}{5}$ $\frac{S = 2}{4.42 \times 10^{-3}}$	$= 10dB$ $S = 3$ 3.15×10^{-3} 3.12×10^{-2} 1.73×10^{-3} $= 5dB$ $S = 3$ 3.13×10^{-3}	S = 4 2.27 × 10 ⁻³ 2.30 × 10 ⁻² 1.20 × 10 ⁻³ $S = 4$ 2.22 × 10 ⁻³
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{tot}}{\xi_{tot}}$ $\frac{\xi_{tot}}{\xi_{int}}$	S = 1 7.85 × 10 ⁻³ 5.37 × 10 ⁻² 5.39 × 10 ⁻³ $S = 1$ 8.38 × 10 ⁻³ 5.66 × 10 ⁻²	SNR = SNR	$= 10dB$ $S = 3$ 3.15×10^{-3} 3.12×10^{-2} 1.73×10^{-3} $= 5dB$ $S = 3$ 3.13×10^{-3} 3.07×10^{-2}	S = 4 2.27 × 10 ⁻³ 2.30 × 10 ⁻² 1.20 × 10 ⁻³ $S = 4$ 2.22 × 10 ⁻³ 2.15 × 10 ⁻²

Table I: *E-shaped Object*, $\tau = 0.10$ - Reconstrution errors: total (ξ_{tot}), internal (ξ_{int}) and external (ξ_{ext}) errors.

		SNR =	= 50 dB	
	S = 1	S=2	S = 3	S = 4
$L^{(S)}$	6.00	3.60	2.10	1.80
$N^{(S)}$	100	208	208	208
$Q^{(S)}$	100	144	49	36
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
$L^{(S)}$	6.00	3.60	2.10	1.80
$N^{(S)}$	100	208	208	208
$Q^{(S)}$	100	144	49	36
~	100	+++	10	00
~	100	SNR =	= 10 dB	
	S = 1	$\frac{SNR}{S=2}$	= 10 dB $S = 3$	S = 4
	S = 1 6.00	SNR = S = 2 3.60	= 10 dB $S = 3$ 2.10	S = 4 1.80
$\frac{L^{(S)}}{N^{(S)}}$	S = 1 6.00 100	SNR = SNR = 3.60 208	= 10 dB = 10 dB S = 3 2.10 208	S = 4 1.80 208
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100	SNR = 2 S = 2 3.60 208 144	= 10 dB $= 10 dB$ $S = 3$ $= 3$ 2.10 $= 208$ $= 49$	S = 4 1.80 208 36
$ \begin{array}{c} L^{(S)} \\ \overline{N^{(S)}} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100	SNR = 2 $S = 2$ 3.60 208 144 $SNR = 2$	$ \begin{array}{r} 10 \\ = 10 \\ B \\ S = 3 \\ 2.10 \\ 208 \\ 49 \\ = 5 \\ dB \end{array} $	S = 4 1.80 208 36
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100 S = 1	SNR = 2 S = 2 3.60 208 144 SNR = 2	= 10dB $= 10dB$ $S = 3$ $= 3$ 2.10 208 49 $= 5dB$ $S = 3$	S = 4 1.80 208 36 S = 4
$ \begin{array}{c} L^{(S)} \\ \overline{D}^{(S)} \\ $	S = 1 6.00 100 100 $S = 1$ 6.00	SNR = 2 S = 2 3.60 208 144 SNR = 2 3.60	= 10dB $= 10dB$ $S = 3$ $= 3$ 2.10 208 49 $= 5dB$ $S = 3$ 2.40	S = 4 1.80 208 36 S = 4 2.10
$ \begin{array}{c} L^{(S)} \\ \overline{N^{(S)}} \\ \overline{Q^{(S)}} \\ \hline L^{(S)} \\ \overline{N^{(S)}} \\ \end{array} $	S = 1 6.00 100 $S = 1$ 6.00 100	SNR = 2 3.60 208 144 SNR = S = 2 3.60 208 208	= 10dB $= 10dB$ $S = 3$ 2.10 208 49 $= 5dB$ $S = 3$ 2.40 208	S = 4 1.80 208 36 $S = 4$ 2.10 208

Table II: *E-shaped Object*, $\tau = 0.10$ - Investigation domain parameters: restricted investigation domain size $L^{(S)}$, total number of cells $N^{(S)}$ and number of cells within the restricted domain size $Q^{(S)}$.





Figure 2: *E-shaped Object*, $\tau = 0.20$ - (*a*) Actual profile and (*b*)-(*o*) IMSA-BCS reconstructed profiles for (*b*)(*e*)(*h*)(*m*) SNR = 20 [dB], (*c*)(*f*)(*i*)(*n*) SNR = 10 [dB] and (*d*)(*g*)(*l*)(*o*) SNR = 5 [dB] at the step (*b*)-(*d*) S = 1, (*e*)-(*g*) S = 2, (*h*)-(*l*) S = 3 and (*m*)-(*o*) S = 4.

		SNR =	= 50 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	1.57×10^{-2}	8.80×10^{-3}	6.46×10^{-3}	4.23×10^{-3}
ξ_{int}	8.62×10^{-2}	8.46×10^{-2}	6.11×10^{-2}	3.86×10^{-2}
ξ_{ext}	1.17×10^{-2}	4.91×10^{-3}	3.70×10^{-3}	2.40×10^{-3}
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	1.60×10^{-2}	8.89×10^{-3}	6.59×10^{-3}	4.82×10^{-3}
ξ_{int}	8.85×10^{-2}	8.55×10^{-2}	6.13×10^{-2}	4.38×10^{-2}
ξ_{ext}	1.19×10^{-2}	4.96×10^{-3}	3.80×10^{-3}	2.81×10^{-3}
30000				
		SNR =	= 10 <i>dB</i>	
	S = 1	SNR = S = 2	= 10dB $S = 3$	S = 4
ξtot	S = 1 1.65×10^{-2}	$SNR = $ $S = 2$ 8.45×10^{-3}	= $10dB$ S = 3 6.50×10^{-3}	S = 4 4.45×10^{-3}
ξ_{tot} ξ_{int}	$\frac{S = 1}{1.65 \times 10^{-2}}$ 9.33 × 10 ⁻²	$SNR = 0$ $S = 2$ 8.45×10^{-3} 7.78×10^{-2}	= $10dB$ S = 3 6.50×10^{-3} 6.03×10^{-2}	$S = 4 4.45 \times 10^{-3} 4.03 \times 10^{-2}$
$\frac{\xi_{tot}}{\xi_{int}}$	S = 1 1.65 × 10 ⁻² 9.33 × 10 ⁻² 1.19 × 10 ⁻²	$SNR = S = 2 8.45 \times 10^{-3} 7.78 \times 10^{-2} 4.72 \times 10^{-3}$	= 10dB S = 3 6.50×10^{-3} 6.03×10^{-2} 3.80×10^{-3}	$S = 4 4.45 \times 10^{-3} 4.03 \times 10^{-2} 2.55 \times 10^{-3}$
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{ext}}$	$S = 1$ 1.65×10^{-2} 9.33×10^{-2} 1.19×10^{-2}	$SNR = SNR = SNR = SNR = SNR = SA5 \times 10^{-3} = 7.78 \times 10^{-2} = 4.72 \times 10^{-3} = SNR = SN$	= 10dB S = 3 6.50×10^{-3} 6.03×10^{-2} 3.80×10^{-3} = 5dB	$S = 4 4.45 \times 10^{-3} 4.03 \times 10^{-2} 2.55 \times 10^{-3}$
$\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 1.65×10^{-2} 9.33×10^{-2} 1.19×10^{-2} $S = 1$	$SNR = SNR = SNR = SNR = SA5 \times 10^{-3}$ 7.78×10^{-2} 4.72×10^{-3} $SNR = SNR =$	$= 10dB$ $S = 3$ 6.50×10^{-3} 6.03×10^{-2} 3.80×10^{-3} $= 5dB$ $S = 3$	$S = 4$ 4.45×10^{-3} 4.03×10^{-2} 2.55×10^{-3} $S = 4$
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{tot}}$	$S = 1$ 1.65×10^{-2} 9.33×10^{-2} 1.19×10^{-2} $S = 1$ 1.76×10^{-2}	SNR = SNR = SNR = SAS + SAS	$= 10dB$ $S = 3$ 6.50×10^{-3} 6.03×10^{-2} 3.80×10^{-3} $= 5dB$ $S = 3$ 6.40×10^{-3}	$S = 4$ 4.45×10^{-3} 4.03×10^{-2} 2.55×10^{-3} $S = 4$ 4.62×10^{-3}
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{tot}}{\xi_{tot}}$ $\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 1.65×10^{-2} 9.33×10^{-2} 1.19×10^{-2} $S = 1$ 1.76×10^{-2} 9.51×10^{-2}	$SNR = SNR = SNR = SNR = SA5 \times 10^{-3}$ 7.78×10^{-2} 4.72×10^{-3} $SNR = SNR =$	$= 10dB$ $S = 3$ 6.50×10^{-3} 6.03×10^{-2} 3.80×10^{-3} $= 5dB$ $S = 3$ 6.40×10^{-3} 5.92×10^{-2}	$S = 4$ 4.45×10^{-3} 4.03×10^{-2} 2.55×10^{-3} $S = 4$ 4.62×10^{-3} 4.36×10^{-2}

Table III: *E-shaped Object*, $\tau = 0.20$ - Reconstrution errors: total (ξ_{tot}), internal (ξ_{int}) and external (ξ_{ext}) errors.

		SNR =	= 50 dB	
	S = 1	S=2	S = 3	S = 4
$L^{(S)}$	6.00	2.10	1.80	1.80
$N^{(S)}$	100	208	208	208
$Q^{(S)}$	100	144	49	36
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
$L^{(S)}$	6.00	2.10	1.80	1.80
$N^{(S)}$	100	208	208	208
O(S)	100	144	49	36
Q`´	100	111	10	00
Q	100	SNR =	= 10 dB	00
	S = 1	$\frac{SNR}{S=2}$	= 10 dB $S = 3$	S = 4
	S = 1 6.00	SNR = 2 $S = 2$ 2.10	= 10 dB $= 3$ 1.80	S = 4 1.80
$\begin{array}{c} L^{(S)} \\ N^{(S)} \end{array}$	S = 1 6.00 100	SNR = 2 $S = 2$ 2.10 208	= 10 dB = 10 dB S = 3 1.80 208	S = 4 1.80 208
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100	SNR = SNR = 2 2.10 208 144	= 10 dB $= 10 dB$ $S = 3$ 1.80 208 49	S = 4 1.80 208 36
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100	SNR = 2 $S = 2$ 2.10 208 144 $SNR = 2$	$ \begin{array}{r} 10 \\ = 10 \\ B \\ S = 3 \\ 1.80 \\ 208 \\ 49 \\ = 5 \\ dB \end{array} $	S = 4 1.80 208 36
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \end{array} $	S = 1 6.00 100 100 S = 1	SNR = 2 S = 2 2.10 208 144 SNR = 2	= 10dB $= 10dB$ $S = 3$ 1.80 208 49 $= 5dB$ $S = 3$	S = 4 1.80 208 36 S = 4
$ \begin{array}{c} L^{(S)} \\ \overline{N^{(S)}} \\ Q^{(S)} \\ \hline L^{(S)} \end{array} $	S = 1 6.00 100 $S = 1$ 6.00	SNR = 2 2.10 208 144 $SNR = 2$ 2.10 2.10 2.10 2.10 2.10 2.10 2.10 2.10	= 10dB $= 10dB$ $S = 3$ 1.80 208 49 $= 5dB$ $S = 3$ 1.80	S = 4 1.80 208 36 S = 4 1.80
$ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ Q^{(S)} \\ \end{array} $ $ \begin{array}{c} L^{(S)} \\ N^{(S)} \\ \end{array} $	S = 1 6.00 100 $S = 1$ 6.00 100 100	SNR = 2 2.10 208 144 SNR = S = 2 2.10 2.10 2.10 2.10 2.10 2.10 2.10 2.1	= 10dB $= 10dB$ $S = 3$ 1.80 208 49 $= 5dB$ $S = 3$ 1.80 208	S = 4 1.80 208 36 $S = 4$ 1.80 208

Table IV: *E-shaped Object*, $\tau = 0.20$ - Investigation domain parameters: restricted investigation domain size $L^{(S)}$, total number of cells $N^{(S)}$ and number of cells within the restricted domain size $Q^{(S)}$.



Figure 3: *E-shaped Object*, $\tau = 0.50$ - (a) Actual profile and (b)-(o) IMSA-BCS reconstructed profiles for (b)(e)(h)(m) SNR = 20 [dB], (c)(f)(i)(n) SNR = 10 [dB] and (d)(g)(l)(o) SNR = 5 [dB] at the step (b)-(d) S = 1, (e)-(g) S = 2,(h)-(l) S - 3 and (m)-(o) S

		SNR =	= 50 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	4.03×10^{-2}	2.25×10^{-2}	1.98×10^{-2}	1.65×10^{-2}
ξ_{int}	1.91×10^{-1}	2.24×10^{-1}	1.97×10^{-1}	1.64×10^{-1}
ξ_{ext}	2.60×10^{-2}	1.15×10^{-2}	1.03×10^{-2}	8.76×10^{-3}
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
ξ_{tot}	3.79×10^{-2}	2.32×10^{-2}	2.14×10^{-2}	1.76×10^{-2}
ξ_{int}	1.90×10^{-1}	2.29×10^{-1}	1.92×10^{-1}	1.76×10^{-1}
ξ_{ext}	2.55×10^{-2}	1.19×10^{-2}	1.18×10^{-2}	9.07×10^{-3}
3000				
3000		SNR =	= 10 <i>dB</i>	
30000	S = 1	SNR = S = 2	= 10dB $S = 3$	S = 4
ξtot	S = 1 4.18×10^{-2}	$SNR = $ $S = 2$ 2.53×10^{-2}	= 10 dB S = 3 1.96×10^{-2}	S = 4 1.72×10^{-2}
ξ_{tot} ξ_{int}	$S = 1 4.18 \times 10^{-2} 2.06 \times 10^{-1}$	$SNR = 2 \\ S = 2 \\ 2.53 \times 10^{-2} \\ 2.22 \times 10^{-1}$	= $10dB$ S = 3 1.96×10^{-2} 1.99×10^{-1}	$S = 4 1.72 \times 10^{-2} 1.72 \times 10^{-1}$
$\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 4.18×10^{-2} 2.06×10^{-1} 2.48×10^{-2}	$SNR = \frac{S = 2}{2.53 \times 10^{-2}}$ $\frac{2.22 \times 10^{-1}}{1.36 \times 10^{-2}}$	= 10dB S = 3 1.96×10^{-2} 1.99×10^{-1} 1.00×10^{-2}	S = 4 1.72 × 10 ⁻² 1.72 × 10 ⁻¹ 8.78 × 10 ⁻³
$\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 4.18×10^{-2} 2.06×10^{-1} 2.48×10^{-2}	$SNR = SNR = 2$ 2.53×10^{-2} 2.22×10^{-1} 1.36×10^{-2} $SNR = 0$	= 10dB S = 3 1.96×10^{-2} 1.99×10^{-1} 1.00×10^{-2} = 5dB	$S = 4$ 1.72×10^{-2} 1.72×10^{-1} 8.78×10^{-3}
$\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 4.18×10^{-2} 2.06×10^{-1} 2.48×10^{-2} $S = 1$	$SNR = \frac{S = 2}{2.53 \times 10^{-2}}$ $\frac{2.22 \times 10^{-1}}{1.36 \times 10^{-2}}$ $SNR = \frac{SNR}{S}$	$= 10dB$ $S = 3$ 1.96×10^{-2} 1.99×10^{-1} 1.00×10^{-2} $= 5dB$ $S = 3$	$S = 4$ 1.72×10^{-2} 1.72×10^{-1} 8.78×10^{-3} $S = 4$
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{tot}}$	$S = 1$ 4.18×10^{-2} 2.06×10^{-1} 2.48×10^{-2} $S = 1$ 4.94×10^{-2}	$SNR = \frac{S = 2}{2.53 \times 10^{-2}}$ 2.22×10^{-1} 1.36×10^{-2} $SNR = \frac{SNR}{2.75 \times 10^{-2}}$	$= 10dB$ $S = 3$ 1.96×10^{-2} 1.99×10^{-1} 1.00×10^{-2} $= 5dB$ $S = 3$ 2.75×10^{-2}	$S = 4$ 1.72×10^{-2} 1.72×10^{-1} 8.78×10^{-3} $S = 4$ 2.75×10^{-2}
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{tot}}{\xi_{tot}}$ $\frac{\xi_{tot}}{\xi_{int}}$	$S = 1$ 4.18×10^{-2} 2.06×10^{-1} 2.48×10^{-2} $S = 1$ 4.94×10^{-2} 2.77×10^{-1}	SNR = SNR	$= 10dB$ $S = 3$ 1.96×10^{-2} 1.99×10^{-1} 1.00×10^{-2} $= 5dB$ $S = 3$ 2.75×10^{-2} 3.33×10^{-1}	$S = 4$ 1.72×10^{-2} 1.72×10^{-1} 8.78×10^{-3} $S = 4$ 2.75×10^{-2} 3.33×10^{-1}

Table V: *E-shaped Object*, $\tau = 0.50$ - Reconstrution errors: total (ξ_{tot}), internal (ξ_{int}) and external (ξ_{ext}) errors.

		SNR =	= 50 dB	
	S = 1	S=2	S=3	S = 4
$L^{(S)}$	6.00	2.40	2.10	2.10
$N^{(S)}$	100	208	208	208
$Q^{(S)}$	100	144	64	49
		SNR =	= 20 dB	
	S = 1	S=2	S = 3	S = 4
$L^{(S)}$	6.00	2.40	2.10	2.10
$N^{(S)}$	100	208	208	208
$Q^{(S)}$	100	144	64	49
		SNR =	= 10 <i>dB</i>	
	S = 1	SNR = S = 2	= 10dB $S = 3$	S = 4
$L^{(S)}$	S = 1 6.00	SNR = 2 2.40	= 10dB $S = 3$ 2.10	S = 4 2.10
$\frac{L^{(S)}}{N^{(S)}}$	S = 1 6.00 100	SNR = 2 $S = 2$ 2.40 208	= 10 dB S = 3 2.10 208	S = 4 2.10 208
$ L^{(S)} \\ N^{(S)} \\ Q^{(S)} $	S = 1 6.00 100 100	SNR = 2 S = 2 2.40 208 144	= 10 dB S = 3 2.10 208 64	S = 4 2.10 208 49
$ \frac{L^{(S)}}{N^{(S)}} $ $ \frac{Q^{(S)}}{Q^{(S)}} $	S = 1 6.00 100 100	SNR = 2 $S = 2$ 2.40 208 144 $SNR = 2$	= 10dB S = 3 2.10 208 64 = 5dB	S = 4 2.10 208 49
$ \frac{L^{(S)}}{N^{(S)}} $ $ Q^{(S)} $	S = 1 6.00 100 100 S = 1	SNR = 2 S = 2 2.40 208 144 SNR = 2	= 10dB S = 3 2.10 208 64 = 5dB S = 3	S = 4 2.10 208 49 S = 4
$L^{(S)}$ $N^{(S)}$ $Q^{(S)}$ $L^{(S)}$	S = 1 6.00 100 100 S = 1 6.00	SNR = 2 2.40 208 144 SNR = 2 1.19	= 10dB $S = 3$ 2.10 208 64 $= 5dB$ $S = 3$ 1.19	S = 4 2.10 208 49 S = 4 1.19
$L^{(S)}$ $N^{(S)}$ $Q^{(S)}$ $L^{(S)}$ $N^{(S)}$	S = 1 6.00 100 100 S = 1 6.00 100	SNR = S = 2 2.40 208 144 SNR = S = 2 1.19 112	= 10dB $S = 3$ 2.10 208 64 $= 5dB$ $S = 3$ 1.19 112	S = 4 2.10 208 49 S = 4 1.19 112

Table VI: *E-shaped Object*, $\tau = 0.50$ - Investigation domain parameters: restricted investigation domain size $L^{(S)}$, total number of cells $N^{(S)}$ and number of cells within the restricted domain size $Q^{(S)}$.



Figure 4: *E-shaped Object*, $\tau = 1.00$ - (a) Actual profile and (b)-(o) IMSA-BCS reconstructed profiles for (b)(e)(h)(m) SNR = 20 [dB], (c)(f)(i)(n) SNR = 10 [dB] and (d)(g)(l)(o) SNR = 5 [dB] at the step (b)-(d) S = 1, (e)-(g) S = 2,(h)-(l) S = 3 and (m)-(o) S

	SNR = 50 dB				
	S = 1	S=2	S=3	S = 4	
ξ_{tot}	6.39×10^{-2}	4.14×10^{-2}	3.35×10^{-2}	3.15×10^{-2}	
ξ_{int}	5.89×10^{-1}	4.29×10^{-1}	4.39×10^{-1}	4.36×10^{-1}	
ξ_{ext}	3.07×10^{-2}	1.66×10^{-2}	1.11×10^{-2}	8.13×10^{-3}	
	S = 1	S=2	S = 3	S = 4	
ξ_{tot}	6.11×10^{-2}	3.96×10^{-2}	3.29×10^{-2}	3.29×10^{-2}	
ξ_{int}	$5.63 imes 10^{-1}$	4.41×10^{-1}	4.11×10^{-1}	4.11×10^{-1}	
ξ_{ext}	$2.78 imes 10^{-2}$	1.43×10^{-2}	$1.10 imes 10^{-2}$	1.10×10^{-2}	
		SNR =	= 10 <i>dB</i>		
	S = 1	SNR = S = 2	= 10dB $S = 3$	S = 4	
ξ _{tot}	S = 1 6.47 × 10 ⁻²	$SNR = $ $S = 2$ 4.68×10^{-2}	= $10dB$ S = 3 4.55×10^{-2}	$S = 4$ 4.47×10^{-2}	
$\frac{\xi_{tot}}{\xi_{int}}$	S = 1 6.47×10^{-2} 5.60×10^{-1}	SNR = 0 S = 2 4.68×10^{-2} 4.43×10^{-1}	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1}	S = 4 4.47×10^{-2} 4.26×10^{-1}	
$\frac{\xi_{tot}}{\xi_{int}}$	S = 1 6.47×10^{-2} 5.60×10^{-1} 3.21×10^{-2}	SNR = S = 2 4.68×10^{-2} 4.43×10^{-1} 2.01×10^{-2}	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1} 1.86×10^{-2}	$S = 4$ 4.47×10^{-2} 4.26×10^{-1} 1.92×10^{-2}	
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{ext}}$	S = 1 6.47×10^{-2} 5.60×10^{-1} 3.21×10^{-2}	SNR = S = 2 4.68×10^{-2} 4.43×10^{-1} 2.01×10^{-2} SNR	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1} 1.86×10^{-2} = $5dB$	$S = 4$ 4.47×10^{-2} 4.26×10^{-1} 1.92×10^{-2}	
$\frac{\xi_{tot}}{\xi_{int}}$	S = 1 6.47×10^{-2} 5.60×10^{-1} 3.21×10^{-2} S = 1	SNR = S = 2 4.68×10^{-2} 4.43×10^{-1} 2.01×10^{-2} SNR S = 2	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1} 1.86×10^{-2} = $5dB$ S = 3	S = 4 4.47×10^{-2} 4.26×10^{-1} 1.92×10^{-2} S = 4	
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{tot}}$	S = 1 6.47×10^{-2} 5.60×10^{-1} 3.21×10^{-2} S = 1 1.29×10^{-1}	SNR = S = 2 4.68×10^{-2} 4.43×10^{-1} 2.01×10^{-2} SNR S = 2 8.10×10^{-2}	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1} 1.86×10^{-2} = $5dB$ S = 3 7.34×10^{-2}	$S = 4$ 4.47×10^{-2} 4.26×10^{-1} 1.92×10^{-2} $S = 4$ 7.44×10^{-2}	
$\frac{\xi_{tot}}{\xi_{int}}$ $\frac{\xi_{ext}}{\xi_{tot}}$	S = 1 6.47×10^{-2} 5.60×10^{-1} 3.21×10^{-2} S = 1 1.29×10^{-1} 5.58×10^{-1}	SNR = SNR	= $10dB$ S = 3 4.55×10^{-2} 4.43×10^{-1} 1.86×10^{-2} = $5dB$ S = 3 7.34×10^{-2} 4.58×10^{-1}	$S = 4$ 4.47×10^{-2} 4.26×10^{-1} 1.92×10^{-2} $S = 4$ 7.44×10^{-2} 4.63×10^{-1}	

Table VII: *E-shaped Object*, $\tau = 1.00$ - Reconstruction errors: total (ξ_{tot}), internal (ξ_{int}) and external (ξ_{ext}) errors.

		SNR = 50 dB				
	S = 1	S=2	S=3	S = 4		
$L^{(S)}$	6.00	2.40	2.10	1.80		
$N^{(S)}$	100	175	175	175		
$Q^{(S)}$	100	100	64	49		
		SNR =	= 20 dB			
	S = 1	S=2	S=3	S = 4		
$L^{(S)}$	6.00	2.10	2.10	2.10		
$N^{(S)}$	100	175	175	175		
$Q^{(S)}$	100	100	49	49		
		SNR =	= 10 dB			
	S = 1	S=2	S=3	S = 4		
$L^{(S)}$	6.00	3.00	2.69	2.69		
$N^{(S)}$	100	208	208	208		
$Q^{(S)}$	100	144	100	81		
		SNR	= 5dB			
	S = 1	S=2	S = 3	S = 4		
$L^{(S)}$	6.00	5.10	4.80	4.50		
$N^{(S)}$	100	343	343	343		
$Q^{(S)}$	100	324	289	256		

Table VIII: *E-shaped Object*, $\tau = 1.00$ - Investigation domain parameters: restricted investigation domain size $L^{(S)}$, total number of cells $N^{(S)}$ and number of cells within the restricted domain size $Q^{(S)}$.

2.1.5 E-shaped Object, $\ell = 1.5\lambda$ - Resume: Errors vs. τ



Figure 5: *E-shaped Object* - Reconstruction errors vs. τ : (*a*) total error, (*b*) internal error and (*c*) external error.





Figure 6: *E-shaped Object* - Reconstruction errors vs. *SNR*: (*a*) total error, (*b*) internal error and (*c*) external error.



Figure 7: *E-shaped Object* - Reconstruction errors vs. *IMSA* step, *S*: (*a*)(*b*) total error, (*c*)(*d*) internal error and (*e*)(*f*) external error for $(a)(c)(e) \tau = 0.1$ and $(b)(d)(f) \tau = 0.5$.



Figure 8: *L-shaped Object*, $\ell = 1.5\lambda$ - Reconstruction errors vs. *IMSA* step, *S*: (*a*)(*b*) total error, (*c*)(*d*) internal error and (*e*)(*f*) external error for (*a*)(*c*)(*e*) *SNR* = 10*dB* and (*b*)(*d*)(*f*) *SNR* = 5*dB*.

More information on the topics of this document can be found in the following list of references.

References

- G. Oliveri, M. Salucci, N. Anselmi, and A. Massa, "Compressive sensing as applied to inverse problems for imaging: theory, applications, current trends, and open challenges," *IEEE Antennas Propag. Mag. - Special Issue on 'Electromagnetic Inverse Problems for Sensing and Imaging*,' vol. 59, no. 5, pp. 34-46, Oct. 2017 (DOI: 10.1109/MAP.2017.2731204).
- [2] A. Massa, P. Rocca, and G. Oliveri, "Compressive sensing in electromagnetics A review," *IEEE Antennas Propag. Mag.*, pp. 224-238, vol. 57, no. 1, Feb. 2015 (DOI: 10.1109/MAP.2015.2397092).
- [3] A. Massa and F. Texeira, "Guest-Editorial: Special Cluster on Compressive Sensing as Applied to Electromagnetics," *IEEE Antennas Wirel. Propag. Lett.*, vol. 14, pp. 1022-1026, 2015 (DOI: 10.1109/LAWP.2015.2425011).
- [4] M. Salucci, L. Poli, F. Zardi, L. Tosi, S. Lusa, and A. Massa, "Contrast source inversion of sparse targets through multi-resolution Bayesian compressive sensing," *Inverse Probl.*, vol. 40, no. 5, p. 055016, May 2024 (DOI 10.1088/1361-6420/ad3b33).
- [5] G. Oliveri, N. Anselmi, M. Salucci, L. Poli, and A. Massa, "Compressive sampling-based scattering data acquisition in microwave imaging," *J. Electromagn. Waves Appl.* J, vol. 37, no. 5, 693-729, Mar. 2023 (DOI: 10.1080/09205071.2023.2188263).
- [6] G. Oliveri, L. Poli, N. Anselmi, M. Salucci, and A. Massa, "Compressive sensing-based Born iterative method for tomographic imaging," *IEEE Tran. Microw. Theory Techn.*, vol. 67, no. 5, pp. 1753-1765, May 2019 (DOI: 10.1109/TMTT.2019.2899848).
- [7] M. Salucci, L. Poli, and G. Oliveri, "Full-vectorial 3D microwave imaging of sparse scatterers through a multi-task Bayesian compressive sensing approach," *Journal of Imaging*, vol. 5, no. 1, pp. 1-24, Jan. 2019 (DOI: 10.3390/jimaging5010019).
- [8] M. Salucci, A. Gelmini, L. Poli, G. Oliveri, and A. Massa, "Progressive compressive sensing for exploiting frequency-diversity in GPR imaging," *J. Electromagn. Waves Appl. J*, vol. 32, no. 9, pp. 1164-1193, 2018 (DOI: 10.1080/09205071.2018.1425160).
- [9] N. Anselmi, L. Poli, G. Oliveri, and A. Massa, "Iterative multi-resolution bayesian CS for microwave imaging," *IEEE Trans. Antennas Propag.*, vol. 66, no. 7, pp. 3665-3677, Jul. 2018 (DOI: 10.1109/TAP.2018.2826574).
- [10] N. Anselmi, G. Oliveri, M. A. Hannan, M. Salucci, and A. Massa, "Color compressive sensing imaging of arbitrary-shaped scatterers," *IEEE Trans. Microw. Theory Techn.*, vol. 65, no. 6, pp. 1986-1999, Jun. 2017 (DOI: 10.1109/TMTT.2016.2645570).
- [11] N. Anselmi, G. Oliveri, M. Salucci, and A. Massa, "Wavelet-based compressive imaging of sparse targets," *IEEE Trans. Antennas Propag.*, vol. 63, no. 11, pp. 4889-4900, Nov. 2015 (DOI: 10.1109/TAP.2015.2444423).

- [12] G. Oliveri, P.-P. Ding, and L. Poli, "3D crack detection in anisotropic layered media through a sparseness-regularized solver," *IEEE Antennas Wirel. Propag. Lett.*, vol. 14, pp. 1031-1034, 2015 (DOI: 10.1109/LAWP.2014.2365523).
- [13] L. Poli, G. Oliveri, P.-P. Ding, T. Moriyama, and A. Massa, "Multifrequency Bayesian compressive sensing methods for microwave imaging," J. Opt. Soc. Am. A, vol. 31, no. 11, pp. 2415-2428, 2014 (DOI: 10.1364/JOSAA.31.002415).
- [14] G. Oliveri, N. Anselmi, and A. Massa, "Compressive sensing imaging of non-sparse 2D scatterers by a total-variation approach within the Born approximation," *IEEE Trans. Antennas Propag.*, vol. 62, no. 10, pp. 5157-5170, Oct. 2014 (DOI: 10.1109/TAP.2014.2344673).