Clustered Linear Arrays with Hilbert-Ordered Excitations

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1 Mathematical Formulation

Let us consider a linear array of N elements equally-spaced by d along the x-axis to be grouped into Q ($Q < N$) sub-arrays, each containing N_q ($q = 1, ..., Q$) elements, so that $\sum_{n=1}^{N} N_q = N$. For beam-forming purposes, each q-th ($q = 1, ..., Q$) sub-array is fed by TRM composed by an amplifier and a phase shifter providing a complex excitation I_q. The array factor of the beam generated at sub-array level is:

$$AF(u) = \sum_{q=1}^{Q} I_q \sum_{n=1}^{N} \delta_{c_n q} e^{jk(n-1)du} \tag{1}$$

where $I_q = \alpha_q e^{j\varphi_q}$, being α_q and φ_q the amplitude and the phase coefficients of the q-th ($q = 1, ..., Q$) sub-array, respectively. Moreover, $k = \frac{2\pi}{\lambda}$ is the wavenumber, λ the wavelength and $u = \sin(\theta)$, being θ the angle measured from broadside. $\delta_{c_n q}$ is the Kronecker delta function:

$$\delta_{c_n} = \begin{cases} 1 & if \ c_n = q \\ 0 & if \ c_n \neq q \end{cases} \tag{2}$$

where $\mathbf{c} = \{c_n \in \mathbb{N} \mid 1 \leq c_n \leq Q, \ n = 1, ..., N\}$ is the integer vector describing the membership of the n-th ($n = 1, ..., N$) array element to the q-th cluster.

Sub-Arraying synthesis problem: given a set of complex excitations, $\{w_n, \ n = 1, ..., N\}$, generating a reference arbitrary-shaped pattern

$$AF^{ref}(u) = \sum_{n=1}^{N} w_n e^{jk(n-1)du} \tag{3}$$

determine the optimal clustering of the array elements into Q disjoint sub-arrays, $\mathbf{c}^{opt} = \{c_n^{opt}, \ n = 1, ..., N\}$ and the values of the complex excitations, $\mathbf{I}^{opt} = \{I_q^{opt}, \ q = 1, ..., Q\}$ so that the beam generated at sub-array level is close ad much as possible to the reference one. For this reason, the synthesis problem is here reformulated as an optimization one in which the *DoFs* (membership vector $\mathbf{c}$, and the sub-array level beamforming excitation vector, $\mathbf{I}$) are set to minimize the following *field-matching* cost function:

$$\Phi(\mathbf{c}, \ \mathbf{I}) = \frac{\int_{-1}^{1} \left| AF^{ref}(u) - AF(u, \ \mathbf{c}, \ \mathbf{I}) \right|^2 du}{\int_{-1}^{1} \left| AF^{ref}(u) \right|^2 du} \tag{4}$$

and this is equivalent to minimize the following *excitation-matching* cost function:

$$\Psi(\mathbf{c}, \ \mathbf{I}) = \frac{1}{N} \sum_{n=1}^{N} \left| w_n - \sum_{q=1}^{Q} \delta_{c_n q} I_q \right|^2 \tag{5}$$

since the remaining terms in (4) are function of neither $\mathbf{c}$ nor $\mathbf{I}$. The optimal values of the q-th ($q = 1, ..., Q$) sub-array coefficient, I_q, for a fixed clustering configuration $\mathbf{c}$ that minimizes (5) is the arithmetic mean of the reference excitations

of the elements belonging to the same q-th ($q = 1, ..., Q$) cluster:

$$I_q\left(\mathbf{c}\right) = \frac{\sum_{n=1}^{N} \delta_{c_n q} w_n}{N_q} \tag{6}$$

Thus, the synthesis can be perfomed only looking to the optimum sub-arraying configuration, $\mathbf{c}^{opt}$, since the sub-array weigths come out as a free through (6).

To evalute the pattern generated with a specific sub-arraying configuration the following another quantity called *power-pattern matching* should be evaluated:

$$\Gamma\left(\mathbf{c},\,\mathbf{I}\right) = \frac{\int_{-1}^{1} \left| \left|AF^{ref}\left(u\right)\right|^2 - \left|AF\left(u,\,\mathbf{c},\,\mathbf{I}\right)\right|^2 \right| du}{\int_{-1}^{1} \left|AF^{ref}\left(u\right)\right|^2 du} \tag{7}$$

For the minimization of $\Psi\left(\mathbf{c}\right)$ an extension of the *contiguous partition method* (*CPM*) has been adopted. It lies within the Fisher's grouping theory, and it requires the creation of an ordered list of values. Once the list is defined the *Border Element Method* (*BEM*) is exploited for efficiently sampling the solution space of the possible clustering configurations (i.e. the contiguous partition of the list of the reference excitations). To compute the ordered list different strategies can be adopted. In this work, the objective is to find a locality-preserving mapping from the 2D complex plane into a one-dimensional space. This techniques have been adopted in computer science for many applications (e.g., range queries, nearest-neighbor queries, clustering and declustering) when multidimensional data is placed into one-dimensional storage (e.g., disk). The idea behind locality-preserving mapping is to map points that are nearby in the multidimensional space into points that are nearby in the one-dimensional space. In the following, two sorting methods will be explained and analysed.

1.1 Hilbert Curve Sorting + BEM

The *Hilbert space-filling curve* (Fig. 1) is a continuous fractal space-filling curve first described by the German mathematician David Hilbert in 1891, as a variant of the space-filling Peano curves discovered by Giuseppe Peano in 1890.
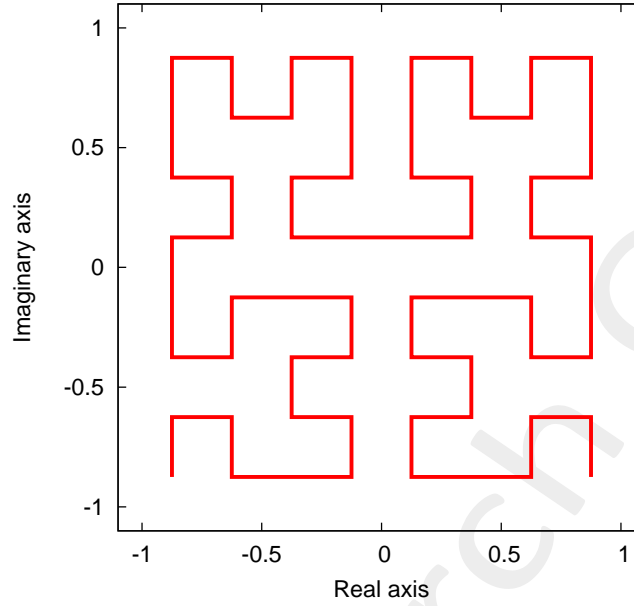

Figure 1: Hilbert Curve - order 3

Space-filling curve is a one-to-one mapping between multidimensional space into one-dimensional space. Among all the space-filling curve, Hilbert curve preserves point neighborhoods *as much as possible*, hence it is widely applied in digital image processing, such image compression, image clustering and object recognition. Hilbert curve has been here chosen to sort the complex excitations of a linear phased array. More in details, to generate a sorted list, the *Hilbert curve* of order H, where H represents the H-th approximation to the limiting curve, has been selected. The clustering procedure works as follows:

- *Step 0 - Hilbert ordered list creation:* given the set of N complex values of the reference vector $\mathbf{w} = \{w_n, \ n = 1, ..., N\}$, define a list $\mathbf{L} = \{\ell_n, \ n = 1, .., N\}$ by ordering the reference excitations according to their position with respect to the Hilbert curve of order H. First of all, a Hilbert curve of order H has been generated to fill a square in the complex plane having dimension $s = 2$ and as center the origin of the plane. Then, each complex excitation w_n, $(n = 1, ..., N)$ has been projected on the curve and the projected excitations have been sorted according to the curve folding, starting from one extreme curve vertex and stopping to the opposite extreme curve vertex. Notice that, if two excitations have the same projection on the Hilbert curve, the first one is the one with the smaller distance from the curve.

- *Step 1 - Initialization:* Once $\mathbf{L}$ is defined, set the initial sub-array configuration $\mathbf{c}^{(t)}$, ($t = 0$, being t the iteration index) by randomly selecting $Q - 1$ cut points among the $N - 1$ admissible ones.

- *Step 2 - BEM application:* apply iteratively the Border Element Method optimization algorithm in order to minimize the *excitation-matching* cost function.

- *Step 3 - Sub-Arrayed Array design:* set $\mathbf{c}^{opt} = \mathbf{c}_{opt}^{(t)}$ and $\mathbf{I}^{opt} = \mathbf{I}_{opt}^{(t)}$.

A variant of the algorithm just described, considers the use of the *Moore Curve* to sort the excitations in the complex plane. In particular, the *Moore curve* is a another continuous fractal space-filling curve. Precisely, it is the loop version of the Hilbert curve, and it may be thought as the union of four copies of the Hilbert curves combined in such a way to make the endpoints coincide (Fig. 2).
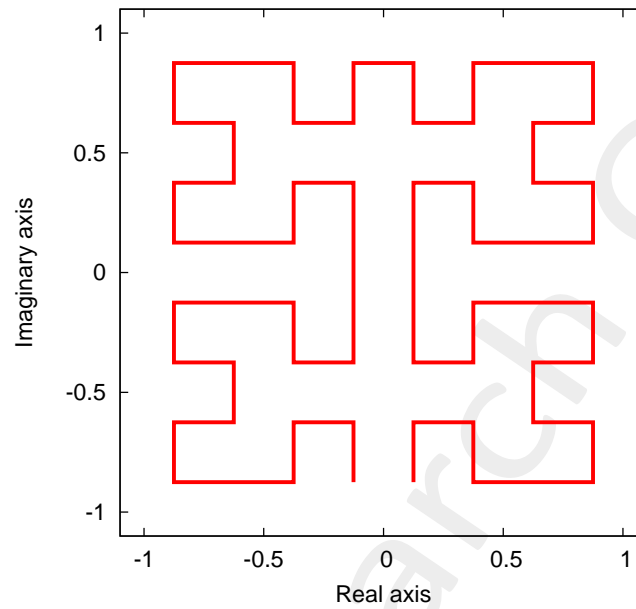

Figure 2: Moore Curve - order 3

1.2 Spectral LPM Sorting + BEM

Fractals divide the space into a number of fragments, visiting the fragments in a specific order. Once a fractal starts to visit points from a certain fragment, no other fragment is visited until the current one is not completely exhausted. By dealing with one fragment, fractals perform a local optimization based on the current fragment. Thus, fractals suffers of the so-called *boundary effect* problem, where points far from the fragment borders are favored. Points that lie near the fragment borders fare the worst, as it can be see from Fig.
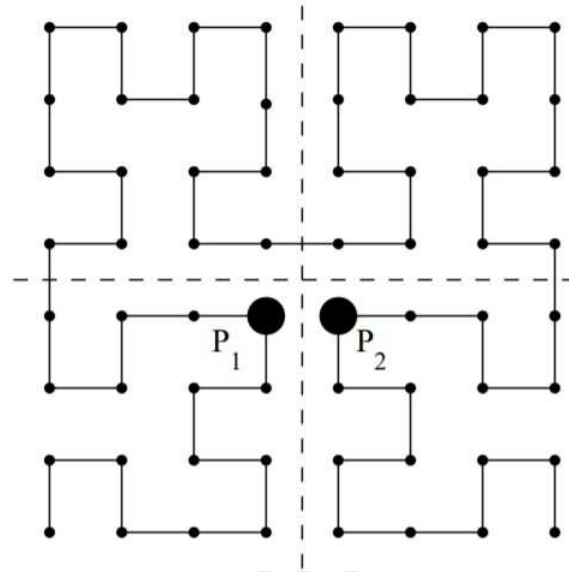
3.

Figure 3: *Boundary Effect:* Hilbert Curve Sorting limitation.

Since points P_1 and P_2 lies on two different quadrants, they are mapped far away from each other. To overcome such a drawback, the *Spectral Locality-Preserving Mapping* (LPM) algorithm has been proposed. Spectral LPM avoids drawbacks of fractals, adopting a global optimization, which means that all the data points in the plane are taken into account when performing the mapping. In general, spectral algorithms use the eigenvalues and eigenvectors of the matrix representation of a graph. In fact, this kind of algorithms has been widely used in graph partitioning and data clustering. For our purpose, a slightly modified version of the algorithm proposed by Mokbel in 2003 has been developed. The main reason of such modification is linked with the type of data to sort. In its work, Mokbelis dealing with memory addresses access (discrete values), while we are dealing with complex excitations (continuous values).

Algorithm 1 *Spectral LPM*

Input: A set of N multi-dimensional points $\mathcal{P} = \{p_i, \ i = 1, ..., N\}$
Output: A linear order $\mathcal{S}$ of the set $\mathcal{P}$

1. Model the set of multi-dimensional points $\mathcal{P}$ as a graph $G(V, \ E)$ such that each point $p_i \in \mathcal{P}$ is represented by a vertex $v_i \in V$, and there is an edge $(v_i, \ v_j) \in E$ if and only if $\|p_i - p_j\| \leq \varepsilon$, being ε a defined threshold.

2. Compute the graph Laplacian matrix, $L(G) = D(G) - A(G)$, where $D(G)$ is the graph diagonal matrix with $D(G)_{ii} = \delta(v_i)$, being $\delta(v_i)$ the degree of the vertex v_i and $A(G)$ is the graph adjacency matrix with $A(G)_{ij} = 1$ if and only if the edge $(i, \ j) \in E$.

3. Compute the second smallest eigenvalue λ_2 and its corresponding eigenvector $X_2 = [x_1, \ x_2, ..., \ x_N]$ of $L(G)$, known as *Fiedler vector*.

4. For each $i = 1 \rightarrow N$, assign the value x_i to v_i and hence to p_i.

5. Sort the eigenvector components in ascending order.

6. The linear order $\mathcal{S}$ of $\mathcal{P}$ is the order of the assigned values of p_i's.

In such a way it is possible to obtain an ordered list of excitations and thus to apply the *Border Element Method* optimization algorithm to minimize the *excitation-matching* cost function (5) and to find the most suitable sub-array configuration. In the following a deeper explanation of the algorithm application will be given and some specific graph concepts will be defined.

Given a set of N complex excitations, $w_n = \Re\{w_n\} + \Im\{w_n\}, \ n = 1, ..., N$, being $\Re\{w_n\}$ and $\Im\{w_n\}$ the real and imaginary part respectively, according to the Spectral LPM algorithm (Alg. 1) an *undirected*, *finite* graph without loops must be computed. Specifically, an *undirected graph* is a graph $G(V, \ E)$, i.e., a set of objects, called vertices or nodes, V, that are connected together by edges E, where all the edges are bidirectional. In contrast, a graph where the edges point in a direction is called a *directed graph*. There are several ways to transform a given set of data points with pairwise similarities or pairwise distances into a graph. In general, two types of graphs can be constructed:

- ε-neighborhood graph: all points, whose pairwise distances are smaller than a given threshold ε, are connected

- k-nearest neighbor graph: vertex v_i is connected with v_j if and only if v_j is among the k-nearest neighbors of v_i, given a similarity measure.

For our aim, a ε-neighborhood graph has been developed and the Euclidean distance has been chosen as similarity measure, i.e. two vertices will be connected if and only if their euclidean distance is less than a defined threshold, ε. To avoid a useless threshold calibration, an analytic formula has been developed:

$$\varepsilon = \max\{\min\{\|w_i - w_j\|\}\}, \ i, \ j = 1, ..., N, \ i \neq j \tag{8}$$

Since each graph vertex v_i could have many edges, to favor smaller distance connections, a *weighted graph* has been used, e.g. a graph with weighted edges. For our purpose, the weights of each edge are defined in the following way:

$$\omega_{ij} = \frac{1}{\|w_i - w_j\|}, \ i, \ j = 1, ..., N, \ i \neq j \tag{9}$$

The proposed algorithm works with a matrix representation of the graph. Therefore, let us define the following matrixes of a graph:

- *Adjacency matrix $A(G)$*: a square matrix used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph, i.e. a (0.1)-matrix with zeros on its diagonal in case of an unweighted graph. Instead, for a weighted graph, it can be defined as a $(0, \omega_{ij})$-matrix. Considering an undirected graph, the matrix is symmetric.

- *Degree matrix $D(G)$*: a diagonal matrix with the degrees $d_1, ..., d_N$ on the diagonal, i.e. the number of edges attached to each vertex.

- *Laplacian matrix $L(G)$*: a symmetric matrix with one row and column for each node defined by:

$$L(G) = A(G) - D(G) \tag{10}$$

The Laplacian matrix has a fundamental role in the *Spectral Graph Theory*, the study of the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrixes associated with the graph, such as its Adjacency matrix or Laplacian matrix. More in details, the eigenvalues of the Laplacian matrix of a graph are closely related to the *connectivity property*.

Definition *(Connected Graph):* a graph is connected if there exists a path between every pair of vertex. In other words, from every vertex to any other vertex, there should be some path to traverse. Instead, a graph with multiple disconnected vertices and edges is called *disconnected.*

Moreover, given an undirected graph and its Laplacian matrix the following proposition holds true:

Proposition: let G be an undirected graph with non-negative weights and $L(G)$ the corresponding Laplacian matrix, then the multiplicity k of the eigenvalue 0 of $L(G)$ equals the number of connected components $B_1, ..., B_k$ in the graph.

As a consequence, a graph is connected if its Laplacian matrix has eigenvalue 0 with multiplicity equal to 1. Analogously, a graph is connected if and only if, given the sorted list of eigenvalues, $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$, the second smallest eigenvalue, $\lambda_2 > 0$. This value represents the algebraic connectivity of a graph. The related eigenvector is known as Fiedler vector and it has been used for graph partitioning.

More information on the topics of this document can be found in the following list of references.

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